Unit- V Complex Variable - Integration

Complex Integral- In case of real variable, the path of integration of $\int_a^b f(x)\,dx$ is always along the x-axis from $x=a$ to $x=b$. But in case of a complex function $f(z)$ the path of the definite integral $\int_a^b f(z)\,dz$ can be along any curve from $z=a$ to $z=b$.

$z = x+iy \Rightarrow dz = dx + i\,dy$ —— ①

Along x-axis $(y=0)$, $dz = dx$ —— ②

Along y-axis $(x=0)$, $dz = i\,dy$ —— ③

In ①, ②, ③ the directions of dz are different. Its value depends upon the path (curve) of integration.

Contour Integral - In case the initial point and final point coincide so that c is a closed curve, then this integral is called contour integral and is denoted by $\oint_c f(z)\,dz$.

Line Integral -

If $f(z) = u(x,y) + iv(x,y)$ then since $dz = dx + i\,dy$

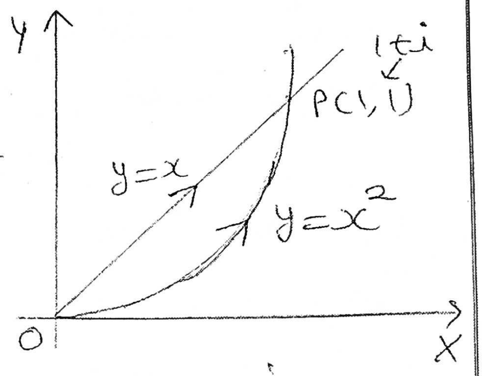
We have,
$$\oint_c f(z)\,dz = \int_c (u+iv)(dx+i\,dy)$$
$$= \int_c (u\,dx - v\,dy) + i\int_c (v\,dx + u\,dy)$$

Which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

Note - If c is a point on the arc joining a and b, then
$$\int_a^b f(z)\,dz = \int_a^c f(z)\,dz + \int_c^b f(z)\,dz$$

Q1. Evaluate $\int_0^{1+i} (x^2 - iy)\,dz$ along the paths

(i) $y = x$ (2018) (ii) $y = x^2$ (2010)



Sol (i) Along the line $y = x \Rightarrow dy = dx$

So that $dz = dx + i\,dy = dx + i\,dx = (1+i)\,dx$

$$\therefore \int_0^{1+i} (x^2 - iy)\,dz = \int_0^1 (x^2 - ix)(1+i)\,dx$$

$$= (1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1$$

$$= (1+i)\left(\frac{1}{3} - i\frac{1}{2}\right) = \frac{1}{3} - \frac{i}{2} + \frac{i}{3} + \frac{1}{2} = \frac{5}{6} - \frac{i}{6}$$

$(i^2 = -1)$

(ii) Along the parabola $y = x^2 \Rightarrow dy = 2x\,dx$

so that $dz = dx + i\,dy = dx + i\,2x\,dx = (1 + 2ix)\,dx$

$$\therefore \int_0^{1+i} (x^2 - iy)\,dz = \int_0^1 (x^2 - ix^2)(1+2ix)\,dx$$

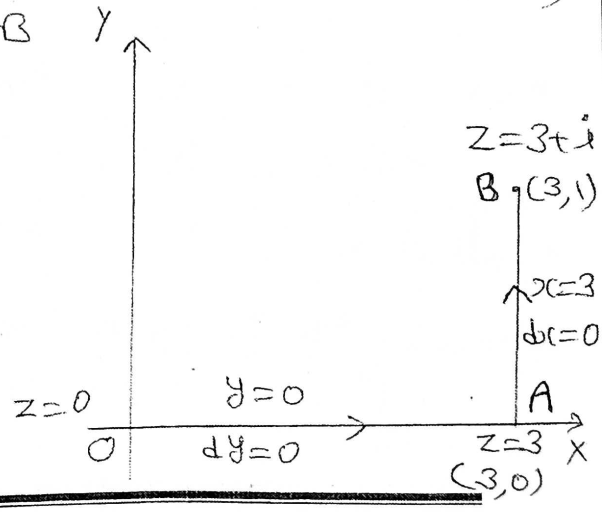
$$= (1-i)\int_0^1 (x^2 + 2ix^3)\,dx = (1-i)\left[\frac{x^3}{3} + 2i\frac{x^4}{4}\right]_0^1$$

$$= (1-i)\left[\frac{1}{3} + \frac{i}{2}\right] = \frac{1}{3} + \frac{i}{2} - \frac{i}{3} + \frac{1}{2} = \frac{5}{6} + \frac{i}{6}$$

Ans

Q2 Evaluate $\int_0^{3+i} (\bar{z})^2\,dz$ along the real axis from $z = 0$ to $z = 3$ and then along a line parallel to Imaginary axis from $z = 3$ to $z = 3+i$ (2013)

Sol. Given path is along OA then AB



$$\therefore \int_0^{3+i} (\bar{z})^2\,dz = \int_{OA} (\bar{z})^2\,dz + \int_{AB} (\bar{z})^2\,dz$$

$(\bar{z})^2 = (x - iy)^2$

Along OA, x varies from 0 to 3.

Along AB, y varies from 0 to 1.

$$\therefore \int_0^{3+i} (\bar{z})^2 dz = \int_{OA} (x-iy)^2 (dx+idy) + \int_{AB} (x-iy)^2 (dx+idy)$$

$$= \int_0^3 x^2 dx + \int_0^1 (3-iy)^2 (idy)$$

$$(\because i^2=-1)$$

$$= \left(\frac{x^3}{3}\right)_0^3 + i\int_0^1 (9-y^2-6iy)dy$$

$$= \frac{27}{3} + i\left[9y - \frac{y^3}{3} - \frac{6iy^2}{2}\right]_0^1$$

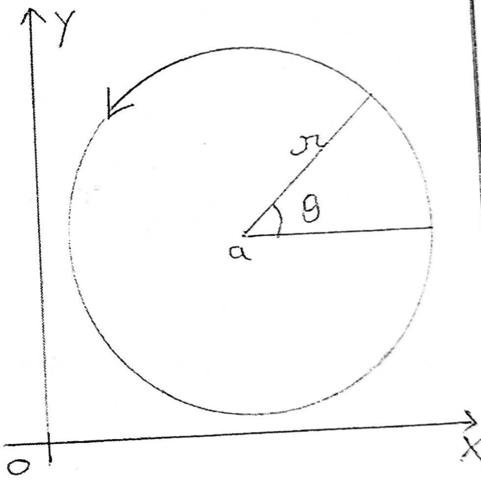
$$= 9 + i\left[9 - \frac{1}{3} - 3i\right] = 9 + \frac{26}{3}i + 3 = 12 + \frac{26}{3}i$$

Ans.

Q3. Prove that (i) $\oint_C \frac{dz}{z-a} = 2\pi i$

(ii) $\oint_C (z-a)^n dz = 0$ [n is an integer $\neq -1$] (2011)

Where C is the circle $|z-a| = r$



Sol: $|z-a| = r \Rightarrow z-a = re^{i\theta}$

$dz = ire^{i\theta} d\theta$

θ varies from 0 to 2π

(i) $$\oint_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta} d\theta}{re^{i\theta}}$$

$$= i\int_0^{2\pi} d\theta = i(\theta)_0^{2\pi} = 2\pi i$$

(ii) $$\oint_C (z-a)^n dz = \int_0^{2\pi} (re^{i\theta})^n \, ire^{i\theta} d\theta$$

$$= ir^{n+1}\int_0^{2\pi} e^{i(n+1)\theta} d\theta = ir^{n+1}\left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi}$$

$$= \frac{ir^{n+1}}{i(n+1)}\left[e^{i(n+1)2\pi} - e^0\right]$$

$(\because e^0 = 1)$

$(\because e^{i\theta} = \cos\theta + i\sin\theta)$

$$= \frac{r^{n+1}}{n+1}\left[\cos(n+1)2\pi + i\sin(n+1)2\pi - 1\right]$$

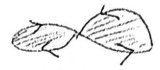
$(\because \sin(n+1)2\pi = 0$ and $\cos(n+1)2\pi = (-1)^{2(n+1)} = 1^{n+1} = 1)$

$$= \frac{r^{n+1}}{n+1}[1+0-1] = 0$$

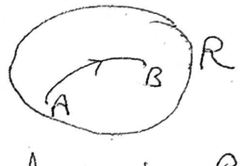
Some Definitions -

① A curve is called <u>simple closed curve</u> if it does not cross itself.

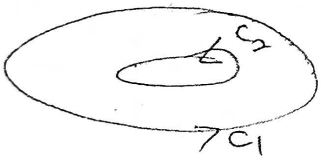
② A curve which crosses itself is called a <u>multiple curve</u>



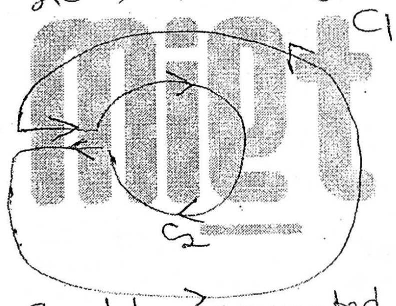
③ A connected region is said to be a <u>simply connected</u> if all the interior points of a closed curve C drawn in the region R are the points of region R.



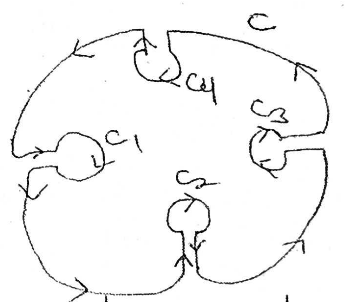
④ <u>Multi-connected region</u> is bounded by more than one curve. We can convert a multi-connected region into a simply connected region, by giving it one or more cuts.



Multi- Connected Region



Simply connected Region



Simply connected Region

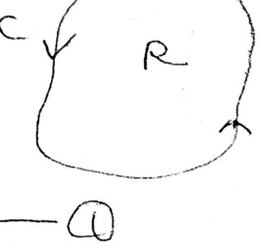
Q1. State and prove Cauchy's Theorem. (2012)

Sol.

Cauchy's Integral Theorem -

<u>Statement-</u> If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a simple closed curve C, then $\oint_C f(z)\,dz = 0$

<u>Proof</u> Let R be the region bounded by the curve C.



Let $f(z) = u(x,y) + iv(x,y)$, then

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+idy)$$

$$= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy) \quad \text{——} ①$$

Since $f'(z)$ is continuous, the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x}$ are also continuous in R. Hence by Green's theorem, we have

$$\oint_C f(z)dz = \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy)$$

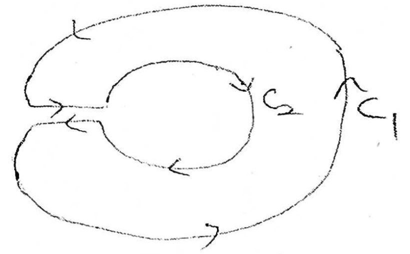
$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \quad \text{---} (2)$$

Now $f(z)$ being analytic at each point of the region R, by C-R equations, we have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence from eq (2) $\oint_C f(z)\,dz = 0 + i(0) = 0$ Proved

Extension of Cauchy's Theorem to Multiply Connected Region-

If $f(z)$ is analytic in the region R between two simple closed curves C_1 and C_2 then



$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

~~where integral along each curve~~

Note- If a closed curve C contains non intersecting closed curves $C_1, C_2, ---, C_n$ then by introducing cross cuts, it can be shown that

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + ---- + \oint_{C_n} f(z)\,dz$$

Q2 Evaluate $\int_{|z|=\frac{1}{2}} \frac{e^z}{z^2+1}\,dz$ (2017)

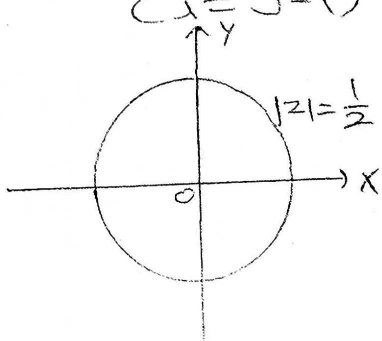
Sol. Singularity (Poles) of given function is where $(i = \sqrt{-1})$

$z^2+1=0 \Rightarrow z^2=-1 \Rightarrow z = \pm i$

$z = i = 0+1i \Rightarrow$ Point $(0,1)$

$z = -i = 0-1i \Rightarrow$ Point $(0,-1)$

$|z| = \frac{1}{2}$ represents circle with centre origin $(0,0)$ and radius $= \frac{1}{2}$

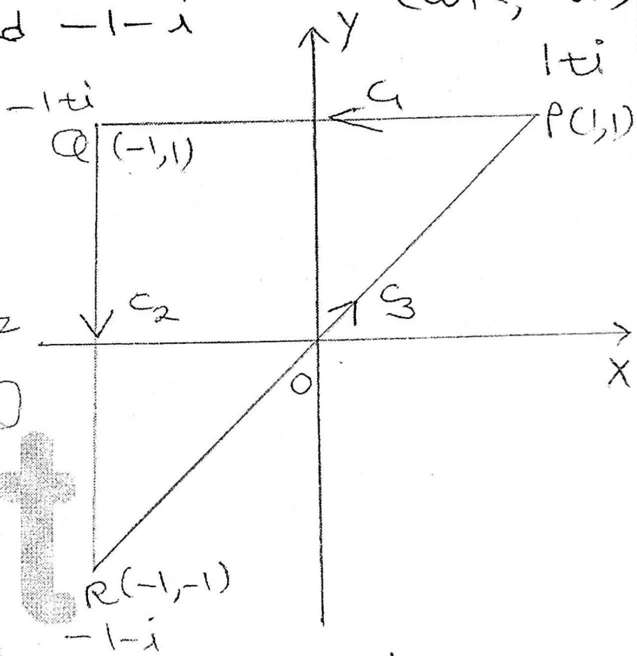


Clearly both the points $z=i$ and $z=-i$ lie outside the circle $|z|=\frac{1}{2}$. Hence function is analytic inside circle. By Cauchy's Theorem

$$\oint_{|z|=\frac{1}{2}} \frac{e^z}{z^2+1}\,dz = 0$$

Ans.

Q3. Verify Cauchy theorem by integrating e^{iz} along the boundary of the triangle with the vertices at the points $1+i$, $-1+i$ and $-1-i$ (2012, 2011)



Sol. The boundary of the triangle C consists of three lines C_1, C_2 and C_3. So

$$\oint_C e^{iz}\,dz = \int_{C_1} e^{iz}\,dz + \int_{C_2} e^{iz}\,dz + \int_{C_3} e^{iz}\,dz \qquad (1)$$

Along C_1, Line PQ is $y=1 \Rightarrow dy=0$

$\therefore\ dz = dx + i\,dy = dx$

x varies from 1 to −1

$$\int_{C_1} e^{iz}\,dz = \int_1^{-1} e^{i(x+i.1)}\,dx = \int_1^{-1} e^{(ix-1)}\,dx = \int_1^{-1} e^{ix}.e^{-1}\,dx$$

$$= e^{-1}\int_1^{-1} e^{ix}\,dx = \frac{1}{e}\left(\frac{e^{ix}}{i}\right)_1^{-1} = \frac{1}{ie}\left(e^{-i}-e^{i}\right) = \frac{1}{i}\left(e^{-1-i}-e^{-1+i}\right)$$

Along C_2, Line QR is $x=-1 \Rightarrow dx=0 \ \therefore\ dz = i\,dy$

$$\int_{C_2} e^{iz}\,dz = \int_1^{-1} e^{i(-1+iy)}\,i\,dy = i\int_1^{-1} e^{-i}\,e^{i^2 y}\,dy = \cancel{\frac{i}{e}\left(\frac{e^{iy}}{i}\right)_1^{-1}}$$

$$= i\,e^{-i}\int_1^{-1} e^{-y}\,dy = i\,e^{-i}\left(-e^{-y}\right)_1^{-1}$$

$$= i\,e^{-i}\left(-e^{+1}+e^{-1}\right) = \frac{i^2}{i}\left(-e^{-i+1}+e^{-1-i}\right)$$

$$= -\frac{1}{i}\left(-e^{-i+1}+e^{-1-i}\right) = \frac{1}{i}\left(e^{1-i}-e^{-1-i}\right)$$

Along C_3, RP Line is $y=x$ $\therefore dy=dx$

So that $dz = dx+idy = dx+idx = (1+i)dx$

$$\int_{C_3} e^{iz}dz = \int_{-1}^{1} e^{i(x+ix)} (1+i)\,dx = (1+i)\int_{-1}^{1} e^{(ix-x)}dx$$

$$= (1+i)\int_{-1}^{1} e^{(i-1)x} dx = (1+i)\left[\frac{e^{(i-1)x}}{i-1}\right]_{-1}^{1}$$

$$= \frac{(1+i)}{\frac{1}{i}(-1-i)}\left[e^{i-1} - e^{-(i-1)}\right]$$

$$= \frac{(1+i)}{(-i)\{-(1+i)\}}\left[e^{i-1} - e^{-i+1}\right] = \frac{1}{i}\left(e^{-1+i} - e^{1-i}\right)$$

From eq. ①

$$\oint_C e^{iz}dz = \frac{1}{i}\left[e^{-1-i} - e^{-1+i} + e^{1-i} - e^{-1-i} + e^{-1+i} - e^{1-i}\right]$$

$$= 0$$

In the triangle PQR there is no singularity, so by Cauchy's theorem $\oint_C e^{iz}dz = 0$

Hence Cauchy's Theorem is verified.

Aliter- (After eq ①)

$$\int_{C_1} e^{iz}dz = \int_{1+i}^{-1+i} e^{iz}dz = \left[\frac{e^{iz}}{i}\right]_{1+i}^{-1+i}$$

$$= \frac{1}{i}\left[e^{i(-1+i)} - e^{i(1+i)}\right] = \frac{1}{i}\left[e^{-i-1} - e^{i-1}\right]$$

$$\int_{C_2} e^{iz}dz = \int_{-1+i}^{-1-i} e^{iz}dz = \left[\frac{e^{iz}}{i}\right]_{-1+i}^{-1-i}$$

$$= \frac{1}{i}\left[e^{i(-1-i)} - e^{i(-1+i)}\right] = \frac{1}{i}\left[e^{-i+1} - e^{-i-1}\right]$$

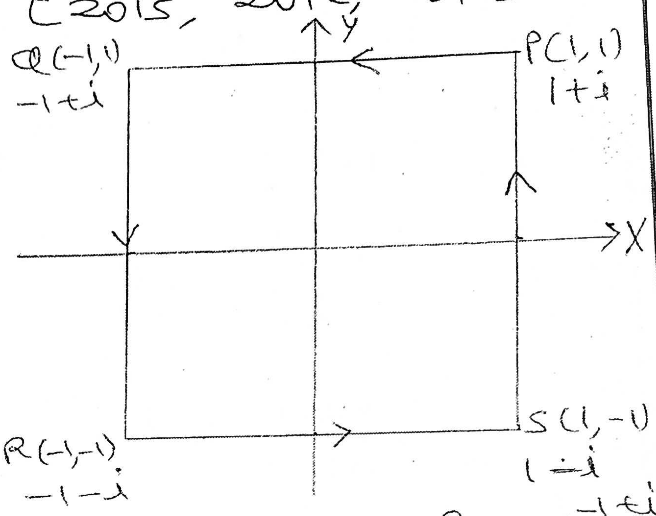
$$\int_{C_3} e^{iz}dz = \int_{-1-i}^{1+i} e^{iz}dz = \left[\frac{e^{iz}}{i}\right]_{-1-i}^{1+i}$$

$$= \frac{1}{i}\left[e^{i(1+i)} - e^{i(-1-i)}\right] = \frac{1}{i}\left[e^{i-1} - e^{-i+1}\right]$$

$$\therefore \int_{C_1} e^{iz}dz + \int_{C_2} e^{iz}dz + \int_{C_3} e^{iz}dz = 0$$

Proved

Q4. Verify Cauchy's theorem for the function $f(z) = 3z^2 + iz - 4$ along the perimeter of the square with vertices $1 \pm i$, $-1 \pm i$. [2015, 2012, 2010]



Sol. In the square PQRS there is no singularity, hence $f(z)$ is analytic in PQRS, so by Cauchy's theorem

$$\int_{PQRS} (3z^2 + iz - 4)\, dz = 0$$

Now square PQRS consists of four lines PQ, QR, RS, SP

$$\int_{PQ} (3z^2 + iz - 4)\, dz = \int_{1+i}^{-1+i} (3z^2 + iz - 4)\, dz = \left[z^3 + \frac{iz^2}{2} - 4z\right]_{1+i}^{-1+i}$$

$$= \left[\left\{(-1+i)^3 + \frac{i}{2}(-1+i)^2 - 4(-1+i)\right\} - \left\{(1+i)^3 + \frac{i}{2}(1+i)^2 - 4(1+i)\right\}\right]$$

$$= \left[(1-i)^3 + \frac{i}{2}(1-i)^2 + 4(1-i) - (1+i)^3 - \frac{i}{2}(1+i)^2 + 4(1+i)\right] \quad \text{---(1)}$$

$$\int_{QR} (3z^2 + iz - 4)\, dz = \int_{-1+i}^{-1-i} (3z^2 + iz - 4)\, dz = \left[z^3 + \frac{iz^2}{2} - 4z\right]_{-1+i}^{-1-i}$$

$$= \left[\left\{(-1-i)^3 + \frac{i}{2}(-1-i)^2 - 4(-1-i)\right\} - \left\{(-1+i)^3 + \frac{i}{2}(-1+i)^2 - 4(-1+i)\right\}\right]$$

$$= \left[-(1+i)^3 + \frac{i}{2}(1+i)^2 + 4(1+i) + (1-i)^3 - \frac{i}{2}(1-i)^2 - 4(1-i)\right] \quad \text{---(2)}$$

$$\int_{RS} (3z^2 + iz - 4)\, dz = \int_{-1-i}^{1-i} (3z^2 + iz - 4)\, dz = \left[z^3 + \frac{iz^2}{2} - 4z\right]_{-1-i}^{1-i}$$

$$= \left[\left\{(1-i)^3 + \frac{i}{2}(1-i)^2 - 4(1-i)\right\} - \left\{(-1-i)^3 + \frac{i}{2}(-1-i)^2 - 4(-1-i)\right\}\right]$$

$$= \left[(1-i)^3 + \frac{i}{2}(1-i)^2 - 4(1-i) + (1+i)^3 - \frac{i}{2}(1+i)^2 - 4(1+i)\right] \quad \text{---(3)}$$

$$\int_{SP} (3z^2 + iz - 4)\, dz = \int_{1-i}^{1+i} (3z^2 + iz - 4)\, dz = \left[z^3 + \frac{iz^2}{2} - 4z\right]_{1-i}^{1+i}$$

$$= \left[\left\{(1+i)^3 + \frac{i}{2}(1+i)^2 - 4(1+i)\right\} - \left\{(1-i)^3 + \frac{i}{2}(1-i)^2 - 4(1-i)\right\}\right]$$

$$= \left[(1+i)^3 + \frac{i}{2}(1+i)^2 - 4(1+i) - (1-i)^3 - \frac{i}{2}(1-i)^2 + 4(1-i)\right] \quad \text{---(4)}$$

Adding eq (1), (2), (3), (4)

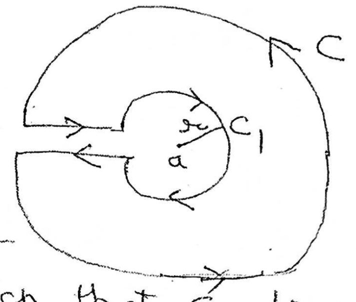
$$\int_{PQ} f(z)\,dz + \int_{QR} f(z)\,dz + \int_{RS} f(z)\,dz + \int_{SP} f(z)\,dz = 0 = \int_{PQRS} f(z)\,dz$$

Hence Cauchy's Theorem verified.

Cauchy's Integral Formula

Statement- If $f(z)$ is analytic within and on a closed curve C and a is any point within C, then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz$$



Proof- Consider the function $\frac{f(z)}{z-a}$, which is analytic at every point within C except at $z=a$. Draw a circle C_1 with centre at a and radius r such that C_1 lies entirely inside C. Thus $\frac{f(z)}{z-a}$ is analytic in the region between C and C_1.

$\therefore$ By Cauchy's theorem, we have

$$\oint_C \frac{f(z)}{z-a}\, dz = \oint_{C_1} \frac{f(z)}{z-a}\, dz \quad \text{--- (1)}$$

Now the equation of circle C_1 is

$|z-a| = r$ or $z-a = re^{i\theta} \Rightarrow dz = ire^{i\theta}\, d\theta$

$$\therefore \oint_{C_1} \frac{f(z)}{z-a}\, dz = \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\, d\theta = i\int_0^{2\pi} f(a+re^{i\theta})\, d\theta$$

Hence from eq (1),

$$\oint_C \frac{f(z)}{z-a}\, dz = i\int_0^{2\pi} f(a+re^{i\theta})\, d\theta \quad \text{--- (2)}$$

In the limiting form, as the circle C_1 shrinks to the point a, i.e., $r \to 0$, then from (2),

$$\oint_C \frac{f(z)}{z-a}\, dz = i\int_0^{2\pi} f(a)\, d\theta = i\, f(a)\, [\theta]_0^{2\pi} = 2\pi i\, f(a)$$

Hence
$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\, dz$$

Proved

For questions Cauchy's Integral Formula is used as

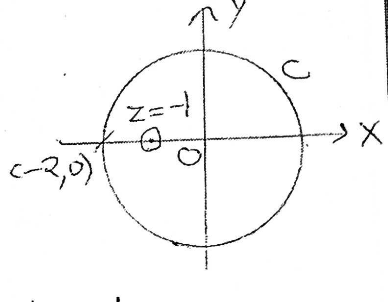
$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \Rightarrow \boxed{\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)}$$

Q1. Evaluate $\int_C \frac{e^z}{z+1}\,dz$, where C is the circle $|z|=2$. (2017)

Sol. $f(z)=e^z$ is an analytic function.

Pole is $z+1=0 \Rightarrow z=-1$ (simple pole)

The point $a=-1$ lies inside the circle $|z|=2$ having centre at origin $(0,0)$ and radius 2.



$\therefore$ By Cauchy's integral formula

$$\oint_C \frac{e^z}{z+1}\,dz = 2\pi i\,(e^z)_{z=-1} = 2\pi i e^{-1} = \frac{2\pi i}{e}$$

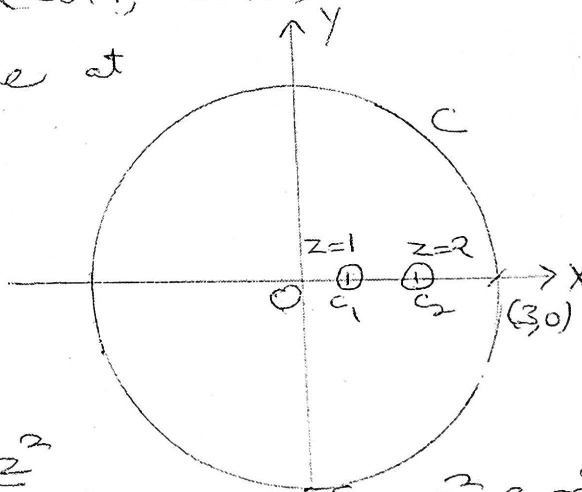
Q2. Use Cauchy Integral formula to evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$, where C is the circle $|z|=3$ (2017, 2010, 2002)

Sol. C is the circle with centre at origin and radius 3.

Poles are given by $(z-1)(z-2)=0$

$\Rightarrow z=1, 2$ (Simple poles)

Both poles lie inside C.



By using Cauchy Integral formula

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = \oint_{c_1} \frac{\frac{\sin \pi z^2 + \cos \pi z^2}{z-2}}{z-1}\,dz + \oint_{c_2} \frac{\frac{\sin \pi z^2 + \cos \pi z^2}{z-1}}{z-2}\,dz$$

$$= 2\pi i\left[\frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\right]_{z=1} + 2\pi i\left[\frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\right]_{z=2}$$

$$= 2\pi i\left[\frac{\sin \pi + \cos \pi}{1-2}\right] + 2\pi i\left[\frac{\sin 4\pi + \cos 4\pi}{2-1}\right]$$

$$= \frac{2\pi i}{(-1)}(0-1) + 2\pi i(0+1) = 2\pi i + 2\pi i = 4\pi i$$

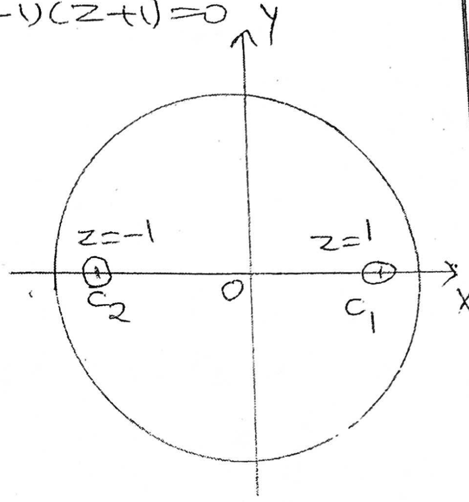
Ans.

Q3. Evaluate $\oint_C \frac{z^2+1}{z^2-1}\,dz$ where c is the circle

(i) $|z| = \frac{3}{2}$ (2018, 2016) (ii) $|z-1| = 1$ (2014) (iii) $|z| = \frac{1}{2}$ (2014)

Sol. Poles are given by $z^2-1=0 \Rightarrow (z-1)(z+1)=0$

$\Rightarrow z = 1, -1$ (Simple Poles)



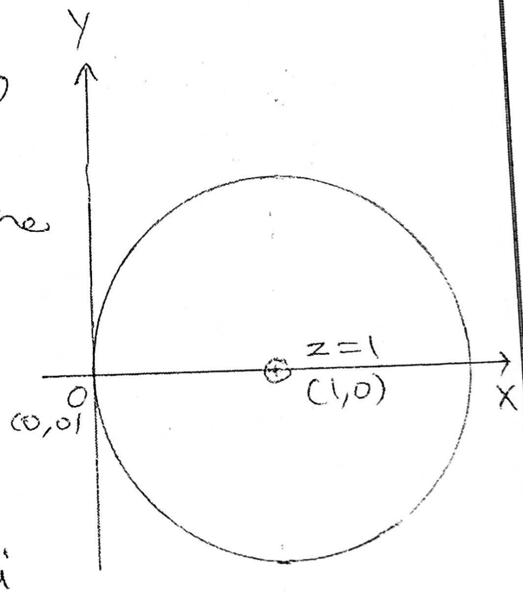
(i) $|z| = \frac{3}{2}$ is the circle with Centre at origin $(0,0)$ & radius $\frac{3}{2}$

Both poles lies inside c.

$$\therefore \oint_C \frac{z^2+1}{z^2-1}\,dz = \oint_{c_1} \frac{\left(\frac{z^2+1}{z+1}\right)}{z-1}\,dz + \oint_{c_2} \frac{\left(\frac{z^2+1}{z-1}\right)}{z+1}\,dz$$

$$= 2\pi i \left(\frac{z^2+1}{z+1}\right)_{z=1} + 2\pi i \left(\frac{z^2+1}{z-1}\right)_{z=-1}$$

$$= 2\pi i \left(\frac{1+1}{1+1}\right) + 2\pi i \left(\frac{1+1}{-1-1}\right) = 2\pi i - 2\pi i = 0$$

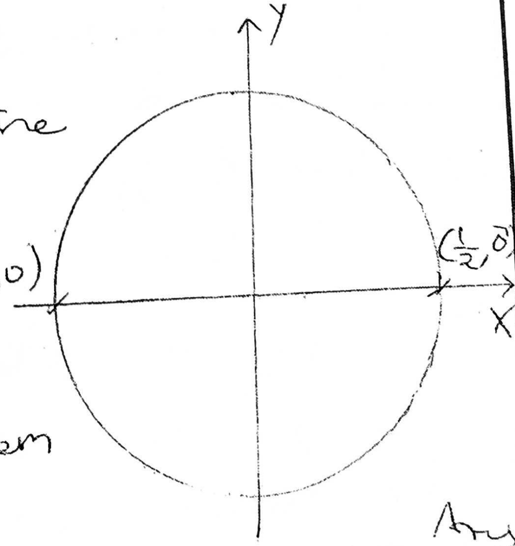


(ii) $|z-1| = 1$ is the circle with centre at $a=1$ i.e. $(1,0)$ and radius 1.

only $z=1$ lie inside $C: |z-1|=1$.

$$\therefore \oint_C \frac{z^2+1}{z^2-1}\,dz = \oint_C \frac{\left(\frac{z^2+1}{z+1}\right)}{z-1}\,dz$$

$$= 2\pi i \left[\frac{z^2+1}{z+1}\right]_{z=1} = 2\pi i \left[\frac{1+1}{1+1}\right] = 2\pi i$$



(iii) $|z| = \frac{1}{2}$ is the circle with centre at origin and radius $\frac{1}{2}$.

clearly both the singularities $z=1$ and $z=-1$ lie outside the circle $|z| = \frac{1}{2}$.

Hence by Cauchy's Integral theorem

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = 0$$

Ans

Cauchy's Integral Formula for the Derivative

If a function $f(z)$ is analytic in a domain D, then at any point $z=a$ of D, $f(z)$ has derivatives of all orders, all of which are again analytic functions in D. their values are given by

$$f^{n}(a) = \frac{\lfloor n}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

where C is any closed contour in D surrounding the point $z=a$.

Note- Above formula can be written as

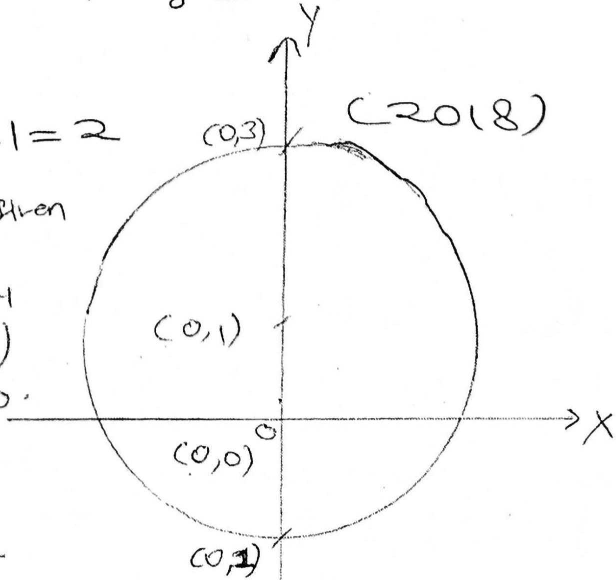
$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{\lfloor n} f^{n}(a)$$

Put; $n=1$, $\displaystyle\int_C \frac{f(z)}{(z-a)^{2}}\,dz = 2\pi i\, f'(a)$

$n=2$, $\displaystyle\int_C \frac{f(z)}{(z-a)^{3}}\,dz = \frac{2\pi i}{\lfloor 2} f''(a)$

$n=3$, $\displaystyle\int_C \frac{f(z)}{(z-a)^{4}}\,dz = \frac{2\pi i}{\lfloor 3} f'''(a)$ and so on.

Q.1. State and prove the Cauchy Integral formula. Also evaluate $\oint_C \frac{1}{(z^2+4)^2}\,dz$ where C is the circle $|z-i|=2$ (2018)



Sol. Poles of $f(z)=\frac{1}{(z^2+4)^2}$ are given by $(z^2+4)^2=0$ $\left(\because z^2+4=0 \Rightarrow z^2=-4,\ z=\pm\sqrt{-4}\right)$

$\Rightarrow z=\pm 2i$ are pole of order two.

$z=2i$ gives point $(0,2)$; inside C

$z=-2i$ gives point $(0,-2)$; outside C

The given curve C is a circle with centre at i i.e. $(0,1)$ and radius 2. Clearly only the pole $z = 2i$ lie inside C while the pole $z = -2i$ lie outside C.

Hence

$$\oint_C \frac{1}{(z^2+4)^2}\,dz = \oint_C \frac{1}{(z+2i)^2(z-2i)^2}\,dz = \oint_C \frac{\frac{1}{(z+2i)^2}}{(z-2i)^2}\,dz$$

$$= 2\pi i\left[\frac{d}{dz}\left\{\frac{1}{(z+2i)^2}\right\}\right]_{z=2i}$$

$$= 2\pi i\left[\frac{-2}{(z+2i)^3}\right]_{z=2i} = \frac{-4\pi i}{(4i)^3} = \frac{-4\pi i}{-64i} = \frac{\pi}{16}$$ Ans

Q.2. Use Cauchy's Integral formula to evaluate $\oint_C \frac{e^{2z}}{(z+1)^4}\,dz$ where C is the circle $|z| = 3$ (2017)

Sol. Poles are given by putting the denominator equal to zero, so $(z+1)^4 = 0 \Rightarrow z = -1$ (Pole of order 4) Which lies within the circle $|z| = 3$

$$\oint_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{\lfloor 3}\left\{\frac{d^3}{dz^3}(e^{2z})\right\}_{z=-1} = \frac{\pi i}{3}\left(8e^{2z}\right)_{z=-1} = \frac{8\pi i}{3e^2}$$ Ans

Q3. State Cauchy's Integral formula. Hence, evaluate $\oint_C \frac{dz}{z^2(z^2-4)e^z}$ Where C is $|z| = 1$ (2013)

Sol Poles are $z = 0$ (double pole) and ~~$z^2 - 4 \Rightarrow z =$~~

$z^2 - 4 = 0 \Rightarrow z = 2, -2$ (Simple pole)

only $z = 0$ lie inside C; circle with centre $(0,0)$ and radius $= 1$. By formula $\oint_C \frac{f(z)}{(z-a)^2}\,dz = 2\pi i f'(a)$

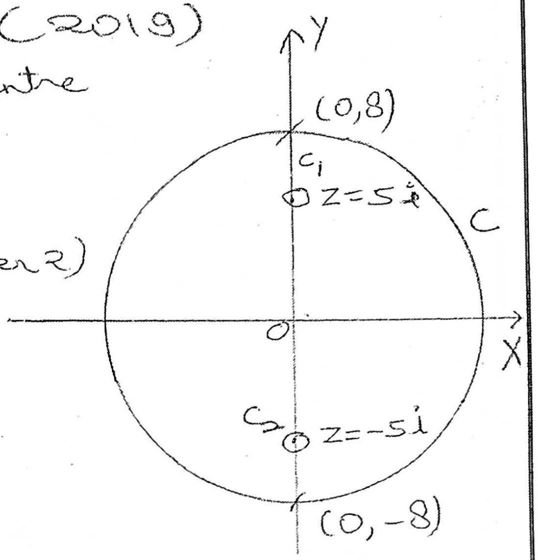
$$\oint_C \frac{dz}{z^2(z^2-4)e^z} = \oint_C \frac{\left(\frac{e^{-z}}{z^2-4}\right)}{z^2}\,dz$$

$$= 2\pi i\left[\frac{d}{dz}\left(\frac{e^{-z}}{z^2-4}\right)\right]_{z=0} = 2\pi i\left[\frac{(z^2-4)(-e^{-z}) - e^{-z}.2z}{(z^2-4)^2}\right]_{z=0}$$

$$= 2\pi i\left[\frac{-4(-1) - 0}{(-4)^2}\right] = 2\pi i\left[\frac{1}{4}\right] = \frac{\pi i}{2}$$ Ans

Q4. Using Cauchy integral formula, evaluate $\int_C \frac{\sin z}{(z^2+25)^2}\,dz$ where C is circle $|z| = 8$. (2019)

Sol. $|z|=8$ is the circle with centre at origin $(0,0)$ and radius $= 8$.



Poles are given by $(z^2+25)^2 = 0$

$\Rightarrow z^2 = -25 \Rightarrow z = \pm 5i$ (pole of order 2)

Both poles lie inside C.

Then using Cauchy Integral formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{\lfloor n}\left[\frac{d^n}{dz^n} f(z)\right]_{z=a}$$

$$\int_C \frac{\sin z}{(z^2+25)^2}\,dz = \int_C \frac{\sin z}{[(z+5i)(z-5i)]^2}\,dz$$

$$= \int_{C_1} \frac{\left[\frac{\sin z}{(z+5i)^2}\right]}{(z-5i)^2}\,dz + \int_{C_2} \frac{\left[\frac{\sin z}{(z-5i)^2}\right]}{(z+5i)^2}\,dz$$

$$= \frac{2\pi i}{\lfloor 1}\left[\frac{d}{dz}\frac{\sin z}{(z+5i)^2}\right]_{z=5i} + \frac{2\pi i}{\lfloor 1}\left[\frac{d}{dz}\frac{\sin z}{(z-5i)^2}\right]_{z=-5i}$$

$$= 2\pi i\left[\frac{(z+5i)^2\cos z - 2\sin z\,(z+5i)}{(z+5i)^4}\right]_{z=5i} + 2\pi i\left[\frac{(z-5i)^2\cos z - 2\sin z\,(z-5i)}{(z-5i)^4}\right]_{z=-5i}$$

$$= 2\pi i\left[\frac{(10i)^2\cos 5i - 2(10i)\sin 5i}{(10i)^4}\right] + 2\pi i\left[\frac{(-10i)^2\cos(-5i) - 2(-10i)\sin(-5i)}{(-10i)^4}\right]$$

$$= \frac{2\pi i}{10^4}\left[-100\cos 5i - 20i\sin 5i\right] + \frac{2\pi i}{10^4}\left[-100\cos 5i - 20i\sin 5i\right]$$

$$= \frac{2\pi i}{10000}\left[-200\cos 5i - 40i\sin 5i\right]$$

$\{\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta\}$

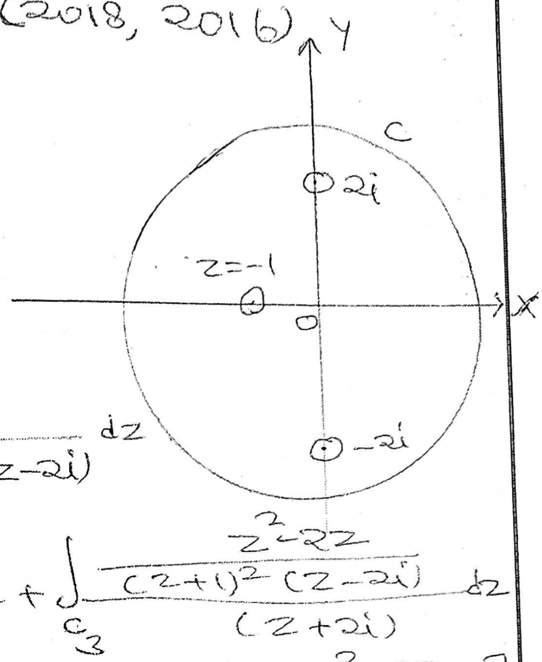
$$= -\frac{40\pi i}{5000}\left[5\cos 5i + i\sin 5i\right]$$

$$= \frac{-\pi i}{125}\left[5\cos 5i + i\sin 5i\right]$$

Ans

Q5. Evaluate by Cauchy integral formula $\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)}dz$
Where C is the circle $|z|=3$.
(2018, 2016)

Sol. $|z|=3$ is the circle with centre at origin and radius 3.



Poles are given by

$(z+1)^2(z^2+4)=0$

$\Rightarrow z=-1$, pole of order 2

$z^2+4=0 \Rightarrow z=\pm 2i$ simple poles.

All poles lie inside C.

$$\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)}dz = \int_C \frac{z^2-2z}{(z+1)^2(z+2i)(z-2i)}dz$$

$$= \int_{C_1} \frac{\frac{z^2-2z}{z^2+4}}{(z+1)^2}dz + \int_{C_2} \frac{\frac{z^2-2z}{(z+1)^2(z+2i)}}{(z-2i)}dz + \int_{C_3} \frac{\frac{z^2-2z}{(z+1)^2(z-2i)}}{(z+2i)}dz$$

$$= 2\pi i\left[\frac{d}{dz}\frac{z^2-2z}{z^2+4}\right]_{z=-1} + 2\pi i\left[\frac{z^2-2z}{(z+1)^2(z+2i)}\right]_{z=2i} + 2\pi i\left[\frac{z^2-2z}{(z+1)^2(z-2i)}\right]_{z=-2i}$$

$$= 2\pi i\left[\frac{(z^2+4)(2z-2)-(z^2-2z).2z}{(z^2+4)^2}\right]_{z=-1} + 2\pi i\left[\frac{-4-4i}{(2i+1)^2(4i)}\right] + 2\pi i\left[\frac{-4+4i}{(-2i+1)^2(-4i)}\right]$$

$$= 2\pi i\left[\frac{(1+4)(-2-2)-(1+2)2(-1)}{(1+4)^2}\right] + \frac{2\pi i(-4)(1+i)}{(-4+1+4i)(4i)} + \frac{2\pi i(-4)(1-i)}{(-4+1-4i)(-4i)}$$

$$= 2\pi i\left[\frac{-20+6}{(5)^2}\right] + \frac{2\pi i(-1)(1+i)}{(-3+4i)\,i} + \frac{2\pi i(1-i)}{(-3-4i)\,i}$$

$$= 2\pi i\left[\frac{-14}{25}\right] - \frac{2\pi i(1+i)}{(-3i-4)} + \frac{2\pi i(1-i)}{(-3i+4)}$$

$$= -\frac{28}{25}\pi i + \frac{2\pi i(1+i)}{(4+3i)} + \frac{2\pi i(1-i)}{(4-3i)}$$

$$= 2\pi i\left[-\frac{14}{25} + \frac{1+i}{4+3i} + \frac{1-i}{4-3i}\right]$$

$$= 2\pi i\left[-\frac{14}{25} + \frac{4-3i+4i+3+4+3i-4i+3}{(4+3i)(4-3i)}\right]$$

$$= 2\pi i\left[\frac{-14}{25} + \frac{14}{16+9}\right] = 2\pi i\left[-\frac{14}{25}+\frac{14}{25}\right] = 0$$

Ans

Taylor's Series (or Theorem)-

If $f(z)$ is analytic inside a circle C with centre at a, then for all z inside C,

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{\lfloor 2} f''(a) + \cdots + \frac{(z-a)^n}{\lfloor n} f^n(a) + \cdots$$

or

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \quad \text{where } a_n = \frac{f^n(a)}{\lfloor n}$$

Laurent's Series (or Theorem)-

If $f(z)$ is analytic inside and on the boundary of the annular (ring shaped) region R bounded by two concentric circles C_1 and C_2 of radii r_1 and r_2 $(r_1 > r_2)$ respectively having centre at a, then for all z in R,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

where $a_n = \dfrac{1}{2\pi i} \oint_{C_1} \dfrac{f(w)}{(w-a)^{n+1}} dw$; $n = 0, 1, 2, \ldots$

and $b_n = \dfrac{1}{2\pi i} \oint_{C_2} \dfrac{f(w)}{(w-a)^{-n+1}} dw$; $n = 1, 2, 3, \ldots$

Note- ① If $f(z)$ is analytic inside C_1, then $b_n = 0$

and $a_n = \dfrac{1}{2\pi i} \oint_{C_1} \dfrac{f(w)}{(w-a)^{n+1}} dw = \dfrac{f^n(a)}{\lfloor n}$

Thus Laurent's series reduces to Taylor's Series

② To expand a fun^c by Laurent theorem is cumbersome. By Binomial theorem, the expansion of a function can be done easily.

③ An analytic function within a circle expanded by Taylor's Series

④ If a function which is not analytic within a circle is expanded by Laurent's series.

Binomial theorem formula - If $|z|<1$, then

$$(1+z)^{-1} = \sum_{n=0}^{\infty} (-1)^n z^n = 1 - z + z^2 - \cdots (-1)^n z^n - \cdots$$

$$(1-z)^{-1} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots + z^n + \cdots$$

Q1. Expand $\frac{1}{z^2-3z+2}$ in the region or

Find Laurent's expansion of $f(z) = \frac{1}{(z-1)(z-2)}$ in the region [2018, 2008]

(a) $|z|<1$ [2019, 2018, 2015, 2010, 2008, 2006]

(b) $1<|z|<2$ [2019, 2018, 2008]

(c) $|z|>2$ [2010]

(d) $0<|z-1|<1$

Sol Here $f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$ (by partial fraction)

(a) When $|z|<1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{-2(1-\frac{z}{2})} + \frac{1}{1-z}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} + (1-z)^{-1} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty} z^n$$

This is a series in positive powers of z, so it is an expansion of $f(z)$ in Taylor's series.

(b) When $1<|z|<2$, i.e. $|z|>1$ and $|z|<2$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{-2(1-\frac{z}{2})} - \frac{1}{z(1-\frac{1}{z})}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

This is a series in positive and negative powers of z, so it is an expansion of $f(z)$ in Laurent's series.

(c) When $|z|>2 \quad \therefore \quad \left|\frac{2}{z}\right|<1$

$$f(z)=\frac{1}{z-2}-\frac{1}{z-1}=\frac{1}{z\left(1-\frac{2}{z}\right)}-\frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$=\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}=\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

This is Laurent's series, since it contains negative powers of z.

(d) When $0<|z-1|<1$

$$f(z)=\frac{1}{z-2}-\frac{1}{z-1}=\frac{1}{(z-1)-1}-\frac{1}{(z-1)}$$

$$=-\frac{1}{1-(z-1)}-\frac{1}{(z-1)}=-[1-(z-1)]^{-1}-\frac{1}{(z-1)}$$

$$=-\sum_{n=0}^{\infty}(z-1)^n-\frac{1}{(z-1)}$$

This is Laurent's series, since it contains a term of negative powers of $(z-1)$.

Q.2. Find the Laurent's expansion for $f(z)=\dfrac{7z-2}{z^3-z^2-2z}$ in the region given by

(i) $0<|z+1|<1$ (ii) $1<|z+1|<3$ (iii) $|z+1|>3$

Sol $z^3-z^2-2z=z(z^2-z-2)=z(z-2)(z+1)$

$$\frac{7z-2}{z^3-z^2-2z}=\frac{7z-2}{z(z-2)(z+1)}=\frac{1}{z}-\frac{3}{z+1}+\frac{2}{z-2}$$

$$=\frac{1}{(z+1)-1}-\frac{3}{(z+1)}+\frac{2}{(z+1)-3}$$

(i) $0<|z+1|<1$

$$f(z)=-[1-(z+1)]^{-1}-\frac{3}{(z+1)}-\frac{2}{3}\left[1-\left(\frac{z+1}{3}\right)\right]^{-1}$$

$$=-\sum_{n=0}^{\infty}(z+1)^n-\frac{3}{z+1}-\frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$$

c(ii) $1 < |z+1| < 3$

$$f(z) = \frac{1}{(z+1)-1} - \frac{3}{(z+1)} + \frac{2}{(z+1)-3}$$

$$= \frac{1}{(z+1)}\left[1 - \frac{1}{z+1}\right]^{-1} - \frac{3}{(z+1)} - \frac{2}{3}\left[1 - \left(\frac{z+1}{3}\right)\right]^{-1}$$

$$= \frac{1}{(z+1)}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n - \frac{3}{z+1} - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$$

c(iii) $|z+1| > 3$

$$f(z) = \frac{1}{(z+1)-1} - \frac{3}{(z+1)} + \frac{2}{(z+1)-3}$$

$$= \frac{1}{(z+1)}\left[1 - \frac{1}{z+1}\right]^{-1} - \frac{3}{(z+1)} + \frac{2}{(z+1)}\left[1 - \frac{3}{z+1}\right]^{-1}$$

$$= \frac{1}{(z+1)}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n - \frac{3}{(z+1)} + \frac{2}{(z+1)}\sum_{n=0}^{\infty}\left(\frac{3}{z+1}\right)^n$$

Since all three expansions contain negative powers of z, they are Laurent's series. Ans.

Q3. Expand $f(z) = \dfrac{z}{(z-1)(2-z)}$ in Laurent's series valid for (i) $|z-1| > 1$ and (ii) $0 < |z-2| < 1$ (2011, 2013, 2014)

Sol By partial fraction

$$\frac{z}{(z-1)(2-z)} = \frac{-z}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{2}{z-2}$$

(i) $|z-1| > 1$

$$f(z) = \frac{1}{(z-1)} - \frac{2}{(z-1)-1} = \frac{1}{(z-1)} - \frac{2}{(z-1)}\left(1 - \frac{1}{z-1}\right)^{-1}$$

$$= \frac{1}{(z-1)} - \frac{2}{(z-1)}\sum_{n=0}^{\infty}\left(\frac{1}{z-1}\right)^n$$

(ii) $0 < |z-2| < 1$

$$f(z) = \frac{1}{z-1} - \frac{2}{z-2} = \frac{1}{(z-2)+1} - \frac{2}{z-2} = [1+(z-2)]^{-1} - \frac{2}{(z-2)}$$

$$= \sum_{n=0}^{\infty}(-1)^n(z-2)^n - \frac{2}{z-2}$$ Ans.

Q4. Expand $\dfrac{(z-2)(z+2)}{(z+1)(z+4)}$ in the regions

(a) $|z|<1$ [2017, 2015], (b) $1<|z|<4$ [2017], (c) $|z|>4$

Sol.

$$f(z)=\frac{(z-2)(z+2)}{(z+1)(z+4)}=\frac{z^2-4}{z^2+5z+4}=\frac{(z^2+5z+4)-5z-8}{z^2+5z+4}$$

$$=1-\frac{5z+8}{(z+1)(z+4)}=1-\frac{1}{z+1}-\frac{4}{z+4}$$

(a) $|z|<1$,

$$f(z)=1-\frac{1}{z+1}-\frac{4}{z+4}=1-(1+z)^{-1}-\frac{4}{4}\left(1+\frac{z}{4}\right)^{-1}$$

$$=1-\sum_{n=0}^{\infty}(-1)^n z^n-\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{4}\right)^n$$

(b) $1<|z|<4$,

$$f(z)=1-\frac{1}{z+1}-\frac{4}{z+4}=1-\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}-\frac{4}{4}\left(1+\frac{z}{4}\right)^{-1}$$

$$=1-\frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n-\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{4}\right)^n$$

(c) $|z|>4$

$$f(z)=1-\frac{1}{z+1}-\frac{4}{z+4}=1-\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}-\frac{4}{z}\left(1+\frac{4}{z}\right)^{-1}$$

$$=1-\frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n-\frac{4}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{4}{z}\right)^n$$ Ans

Q5. Find Taylor's series expansion of $\dfrac{4z-1}{z^4-1}$ about the point $z=0$. [2012]

Sol.

$$f(z)=\frac{4z-1}{z^4-1}=\frac{4z-1}{(z-1)(z+1)(z^2+1)}$$

$$=\frac{A}{z-1}+\frac{B}{z+1}+\frac{Cz+D}{z^2+1}$$

$$\Rightarrow 4z-1 = A(z+1)(z^2+1)+B(z-1)(z^2+1)+(Cz+D)(z-1)(z+1)$$
$$= A(z^3+z^2+z+1)+B(z^3-z^2+z-1)+(Cz+D)(z^2-1)$$
$$= A(z^3+z^2+z+1)+B(z^3-z^2+z-1)+(Cz^3+Dz^2-Cz-D)$$
$$= (A+B+C)z^3+(A-B+D)z^2+(A+B-C)z+(A-B-D)$$

Equating coefficients of z^3, z^2, z and constant terms,

$$A+B+C=0 \quad \text{---} \; (1)$$
$$A-B+D=0 \quad \text{---} \; (2)$$
$$A+B-C=4 \quad \text{---} \; (3)$$
$$A-B-D=-1 \quad \text{---} \; (4)$$

Subtracting eq (1) from (3) $\Rightarrow -2C=4 \Rightarrow C=-2$

Subtracting eq (2) from (4) $\Rightarrow -2D=-1 \Rightarrow D=\frac{1}{2}$

Adding eq (1) & (2) $\Rightarrow 2A+C+D=0 \Rightarrow 2A = 2-\frac{1}{2}=\frac{3}{2} \Rightarrow A=\frac{3}{4}$

From eq (1), $B=-C-A = 2-\frac{3}{4}=\frac{5}{4}$

$$\therefore f(z) = \frac{4z-1}{z^4-1} = \frac{3}{4}\frac{1}{(z-1)}+\frac{5}{4}\frac{1}{(z+1)}+\frac{(-2z+\frac{1}{2})}{z^2+1}$$

Expanding about the point $z=0$, we get

$$f(z) = -\frac{3}{4}(1-z)^{-1}+\frac{5}{4}(1+z)^{-1}+\left(\frac{1}{2}-2z\right)(1+z^2)^{-1}$$
$$= -\frac{3}{4}\sum_{n=0}^{\infty} z^n+\frac{5}{4}\sum_{n=0}^{\infty}(-1)^n z^n+\left(\frac{1}{2}-2z\right)\sum_{n=0}^{\infty}(-1)^n z^{2n}$$

Ans.

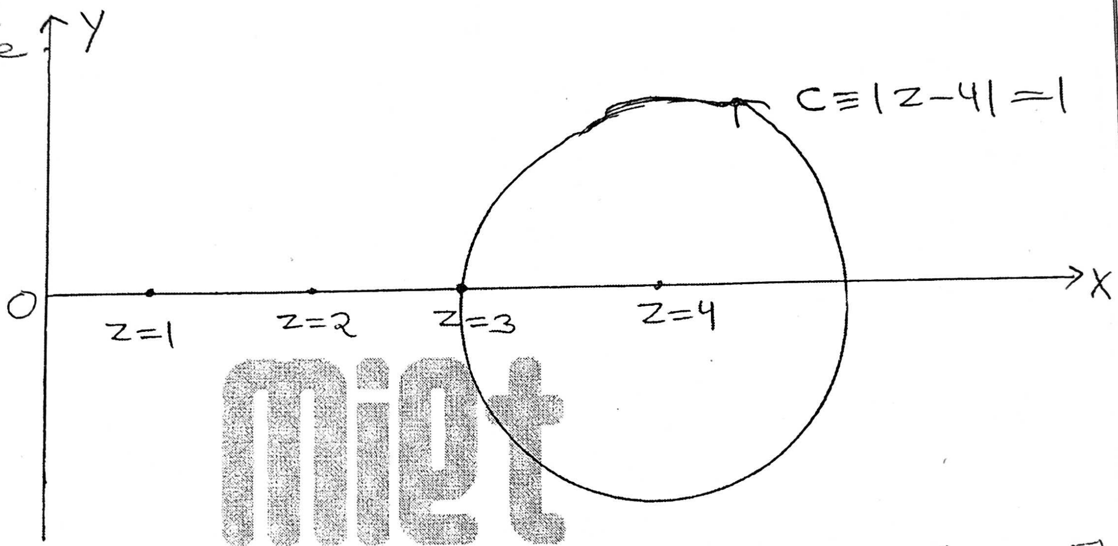
Q6. Obtain the Taylor's series extension of

$f(z) = \dfrac{1}{z^2-4z+3}$ about the point $z=4$.

Find its region of convergence. (2014)

Sol. $$f(z) = \frac{1}{z^2-4z+3} = \frac{1}{(z-1)(z-3)} = \frac{1}{2}\left[\frac{1}{z-3}-\frac{1}{z-1}\right]$$

If the centre of the circle is at $z=4$, then the distances of the singularities $z=1$ and $z=3$ from centre are 3 and 1 respectively. Hence if a circle is drawn with centre at $z=4$ and radius 1 then within a circle $|z-4|=1$, the given function $f(z)$ is analytic hence it can be expanded in Taylor's series within the circle $|z-4|=1$, which is therefore the circle of convergence.



$$f(z) = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right] = \frac{1}{2}\left[\frac{1}{(z-4)+1} - \frac{1}{(z-4)+3}\right]$$

$$= \frac{1}{2}\left[\{1+(z-4)\}^{-1} - \frac{1}{3}\left\{1+\left(\frac{z-4}{3}\right)\right\}^{-1}\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} (-1)^n (z-4)^n - \frac{1}{3}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z-4}{3}\right)^n\right]$$

Ans.

Singular Point or Singularity - A singularity of a function $f(z)$ is a point at which function ceases to be analytic. (i.e. those points where $f(z)$ is not analytic.)

According to position it is of two types -

Isolated and Non-isolated singularity -

If $z=a$ is a singularity of $f(z)$ and there is no other singularity within a small circle surrounding $z=a$, then $z=a$ is called isolated singularity of $f(z)$, otherwise it is called non-isolated singularity.

Ex-1- $f(z) = \dfrac{z+1}{z(z-2)}$

Sol $f(z)$ is analytic everywhere except at $z=0$ and $z=2$ where $f(z) \to \infty$. Since in the neighbourhood of $z=0$ and $z=2$ there are no other singularity. Hence $z=0$ and $z=2$ are isolated singularity.

Ex-2- $f(z) = \cot\left(\frac{\pi}{z}\right)$

Sol $f(z) = \cot\left(\frac{\pi}{z}\right) = \dfrac{1}{\tan\left(\frac{\pi}{z}\right)}$

It is not analytic at the points where

$\tan\left(\frac{\pi}{z}\right) = 0 = \tan n\pi \Rightarrow \frac{\pi}{z} = n\pi \Rightarrow z = \frac{1}{n}$

$n = 1, 2, 3, \ldots$

Thus $z = 1, \frac{1}{2}, \frac{1}{3}, \ldots ; z=0$ are the singularities of $f(z)$

All of which are isolated except $z=0$ because in the neighbourhood of $z=0$, there are infinite number of other singularities $z=\frac{1}{n}$ where $n \to \infty$

Thus $z=0$ is the non-isolated singularity of $f(z)$.

Types of Singularity – By Laurent's theorem

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

The second term $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$ is called the Principal Part (P.P.) of $f(z)$ at the isolated singularity z [illegible] three cases arise

(i) No term in [illegible] $\Rightarrow$ Removable singularity

(ii) Finite number of terms in P.P. $\Rightarrow$ Pole

(iii) Infinite number of terms in P.P. $\Rightarrow$ Essential singularity

Note – The pole is said to be of order n, if there are n terms in the principal part.

Ex 3 – $f(z) = \dfrac{\sin(z-a)}{z-a}$ at $z=a$.

Sol
$$f(z) = \frac{1}{z-a}\left[(z-a) - \frac{(z-a)^3}{\lfloor 3} + \frac{(z-a)^5}{\lfloor 5} - \cdots\right]$$
$$= 1 - \frac{(z-a)^2}{\lfloor 3} + \frac{(z-a)^4}{\lfloor 5} - \cdots$$

Since no terms containing negative powers of $(z-a)$ Hence removable singularity at $z=a$.

Ex. 4 – $f(z) = \dfrac{\sin(z-a)}{(z-a)^4}$ at $z=a$.

Sol
$$f(z) = \frac{1}{(z-a)^4}\left[(z-a) - \frac{(z-a)^3}{\lfloor 3} + \frac{(z-a)^5}{\lfloor 5} - \cdots\right]$$
$$= \frac{1}{(z-a)^3} - \frac{1}{\lfloor 3}\frac{1}{(z-a)} + \frac{1}{\lfloor 5}(z-a) - \cdots$$

Since $f(z)$ has finite number of terms in principal part (first two terms only) i.e. terms of negative powers of $z-a$. Hence $f(z)$ has a pole at $z=a$ of order 2.

Ex.5 - $f(z) = \sin\left(\frac{1}{z-a}\right)$ at $z=a$

Sol $\sin\frac{1}{z-a} = \frac{1}{z-a} - \frac{1}{\lfloor 3}\frac{1}{(z-a)^3} + \frac{1}{\lfloor 5}\frac{1}{(z-a)^5} - -$

Since $f(z)$ has infinite number of terms in P.P.
(i.e. terms of negative powers of $z-a$)
Hence $f(z)$ has essential singularity at $z=a$.

~~Note - ① The limi~~

Zero of an Analytic Function - A zero of an analytic function $f(z)$ is a value of z such that
$f(z) = 0$.

Note - ① The limit point of the zeros of a function $f(z)$ is an isolated essential singularity.

② The limit point of the poles of a function $f(z)$ is a non-isolated essential singularity.

Ex.6 - Find out the zero and discuss the nature of the singularity of $f(z) = \frac{z-2}{z^2}\sin\frac{1}{z-1}$

Sol zeros of $f(z)$ are given by $f(z)=0$

$\Rightarrow z-2=0, \quad \sin\frac{1}{z-1} = 0$

$\Rightarrow z=2, \quad \frac{1}{z-1} = n\pi \Rightarrow z = 1+\frac{1}{n\pi} \quad (n=0,\pm1,\pm2,--)$

clearly $z=1$ is the limit point of zeros (at $n\to\infty$)

Hence $z=1$ is the isolated essential singularity.

Poles of $f(z)$ are given by,
$z^2=0 \Rightarrow z=0$ pole of order 2.

Residue at a Pole

Definition – The coefficient b_1 in the Laurent's series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

is called residue of $f(z)$ at the pole $z=a$. It is denoted by Res. $(z=a)$. Hence

$$\text{Res.}(z=a) = b_1 = \frac{1}{2\pi i}\int_c f(z)\,dz$$

Methods of finding out Residues –

① If $f(z)$ has a simple pole (i.e. pole of order 1) at $z=a$, then

$$\text{Res}[f(z)] = \lim_{z\to a} (z-a) f(z)$$

② If $f(z)$ has a pole of order m at $z=a$, then

$$\text{Res}[f(z)] = \frac{1}{\lfloor m-1} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)]$$

Q1. Find the poles (with its order) and residue at each pole of the following function (2017)

$$f(z) = \frac{1-2z}{z(z-1)(z-2)^2}$$

Sol. Poles of $f(z)$ are given by $z(z-1)(z-2)^2 = 0$

$\Rightarrow$ $z=0$ (simple pole), $z=1$ (simple pole)

and $z=2$ (Pole of order 2)

Res. of $f(z)$ at $z=0$ is, $R_1 = \lim_{z\to 0} z f(z)$

$$= \lim_{z\to 0} \frac{1-2z}{(z-1)(z-2)^2} = \frac{1}{(-1)(-2)^2} = -\frac{1}{4}$$

Residue of $f(z)$ at $z=1$,

$$R_2 = \lim_{z\to 1} (z-1) f(z) = \lim_{z\to 1} \frac{1-2z}{z(z-2)^2} = \frac{1-2}{1(1-2)^2} = -1$$

Residue of $f(z)$ at $z=2$,

$$R_3 = \frac{1}{\lfloor 2-1} \left[\frac{d}{dz} (z-2)^2 f(z)\right]_{z=2} = \left[\frac{d}{dz} \frac{1-2z}{z(z-1)}\right]_{z=2}$$

$$= \left[\frac{(z^2-z)(-2)-(1-2z)(2z-1)}{(z^2-z)^2}\right]_{z=2}$$

$$= \left[\frac{2(-2)-(-3)(3)}{(2)^2}\right] = \left[\frac{-4+9}{4}\right] = \frac{5}{4}$$ Ans.

Q2. Find the residue of $f(z) = \frac{\cos z}{z(z+5)}$ at $z=0$. (2019)

Sol clearly $z=0$ and $z=-5$ are simple poles of $f(z)$.

Res. $f(z)$ at $z=0$ is $\lim_{z\to 0} z \cdot \frac{\cos z}{z(z+5)} = \lim_{z\to 0} \frac{\cos z}{z+5} = \frac{1}{5}$

Additional Questions on Residue

Case I – If $f(z)$ is of the form $f(z) = \frac{\phi(z)}{\psi(z)}$; $\psi(a)=0$

and $\phi(a) \neq 0$ where $z=a$ is the simple pole of $f(z)$

then residue of $f(z)$ at $z=a$ is $\frac{\phi(a)}{\psi'(a)}$.

Q3. Determine the poles and residue at each pole of the function $f(z) = \cot z$ (2017, 2016)

Sol $f(z) = \cot z = \frac{\cos z}{\sin z}$

Poles of $f(z)$ are given by

$\sin z = 0 \Rightarrow z = n\pi$, where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

Residue of $f(z)$ at $z=n\pi$ is $= \left[\frac{\cos z}{\frac{d}{dz}(\sin z)}\right]_{z=n\pi}$

$$= \left[\frac{\cos z}{\cos z}\right]_{z=n\pi} = 1$$ Ans.

Q4. Determine the poles of the function and residue at the poles $f(z) = \frac{z}{\sin z}$

Sol Poles of $f(z)$ are $\sin z = 0 \Rightarrow z = n\pi$; $n = 0, \pm 1, \pm 2, \ldots$

Residue $= \left(\frac{z}{\cos z}\right)_{z=n\pi} = \frac{n\pi}{\cos n\pi} = \frac{n\pi}{(-1)^n}$ Ans

Case II - Residue of $f(z)$ at $z=a$ (Simple pole or pole of order m) is equals to the coefficient of $\frac{1}{t}$ in $f(a+t)$ expansion.

Q5. Find the residue at $z=0$ of $z\cos\frac{1}{z}$

Sol. $z\cos\frac{1}{z} = z\left[1 - \frac{1}{\lfloor 2\, z^2} + \frac{1}{\lfloor 4\, z^4} - \cdots\right] = z - \frac{1}{2z} + \frac{1}{\lfloor 4\, z^3} - \cdots$

The coefficient of $\frac{1}{z}$ in it is $-\frac{1}{2}$.

So the residue of $z\cos\frac{1}{z}$ at $z=0$ is $-\frac{1}{2}$.

Case III - Residue of $f(z)$ at $z=\infty$ is $= \lim_{z\to\infty}[-zf(z)]$

or $= -$ [Coefficient of $\frac{1}{z}$ in expansion of $f(z)$]

Q6. Find the residue of $f(z) = \frac{z^3}{z^2-1}$ at $z=\infty$

Sol $\because \lim_{z\to\infty}\left[-z.\frac{z^3}{z^2-1}\right]$ does not exist.

Therefore $f(z) = \frac{z^3}{z^2\left(1-\frac{1}{z^2}\right)} = z\left(1-\frac{1}{z^2}\right)^{-1}$

$= z\left(1 + \frac{1}{z^2} + \frac{1}{z^4} + \cdots\right) = z + \frac{1}{z} + \frac{1}{z^3} + \cdots$

Residue $= -(\text{Coeff. of } \frac{1}{z}) = -1$ Ans

Cauchy's Residue Theorem (or the Theorem of Residues)

Statement Let $f(z)$ be single valued and analytic within and on a closed contour C except at a finite number of poles $z_1, z_2, ---, z_n$ and let $R_1, R_2, ---, R_n$ be respectively the residues of $f(z)$ at these poles

Then $$\oint_C f(z)\,dz = 2\pi i\,(R_1 + R_2 + \text{---} + R_n)$$

Proof Let $C_1, C_2, ---, C_n$ be the circles with centres at $z_1, z_2, ---, z_n$ respectively and radii so small that they lie entirely within the closed curve C and do not overlap. Then $f(z)$ is analytic within the region enclosed by the curve C and these circles. Hence by Cauchy's theorem for multi-connected region, we have

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \text{----} + \int_{C_n} f(z)\,dz \qquad \text{---} (1)$$

But by the definition of residue

$$R_1 = \frac{1}{2\pi i}\int_{C_1} f(z)\,dz \Rightarrow \int_{C_1} f(z)\,dz = 2\pi i R_1$$

similarly $$\int_{C_2} f(z)\,dz = 2\pi i R_2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\int_{C_n} f(z)\,dz = 2\pi i R_n$$

Hence from eq (1),

$$\int_C f(z)\,dz = 2\pi i R_1 + 2\pi i R_2 + \text{----} + 2\pi i R_n$$

$$= 2\pi i\,(R_1 + R_2 + \text{----} + R_n)$$

Proved

Q1. Determine the poles and residues at each pole for $\dfrac{z-1}{(z+1)^2(z-2)}$ and hence evaluate $\oint_C f(z)\,dz$, where C is the circle $|z-i|=2$ (2014)

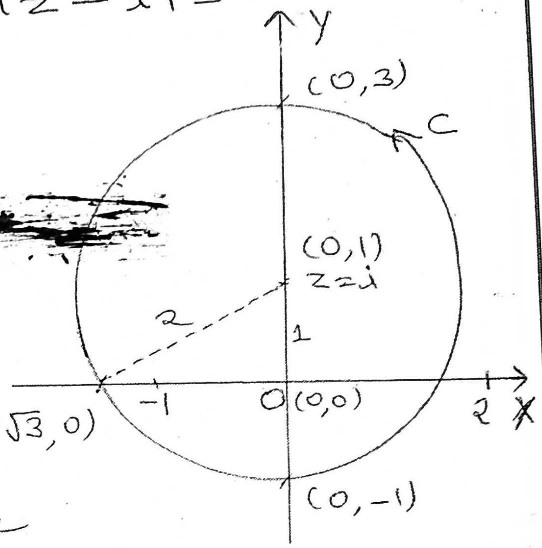
Sol. Poles of $f(z) = \dfrac{z-1}{(z+1)^2(z-2)}$

are given by

$(z+1)^2(z-2)=0$

$\Rightarrow z=-1$ (double pole)

and $z=2$ (simple pole)



$|z-i|=2$ is the circle with centre at i i.e. at $(0,1)$ and radius 2. Clearly, only the pole $z=-1$ lies inside the circle C.

Residue of $f(z)$ at $z=-1$ is $R_1 = \dfrac{1}{\lfloor 2-1}\left[\dfrac{d}{dz}\left\{(z+1)^2\dfrac{z-1}{(z+1)^2(z-2)}\right\}\right]_{z=-1}$

$R_1 = \left[\dfrac{d}{dz}\left(\dfrac{z-1}{z-2}\right)\right]_{z=-1} = \left[\dfrac{d}{dz}\left(1+\dfrac{1}{z-2}\right)\right]_{z=-1} = \left[-\dfrac{1}{(z-2)^2}\right]_{z=-1} = -\dfrac{1}{(-3)^2} = -\dfrac{1}{9}$

Residue of $f(z)$ at $z=2$ is $R_2 = \lim\limits_{z\to 2}(z-2)\dfrac{(z-1)}{(z+1)^2(z-2)}$

$R_2 = \lim\limits_{z\to 2}\dfrac{z-1}{(z+1)^2} = \dfrac{2-1}{(3)^2} = \dfrac{1}{9}$

Now

$\oint_C f(z)\,dz = 2\pi i\,R_1 = 2\pi i\left(-\dfrac{1}{9}\right) = -\dfrac{2\pi i}{9}$

Q2 Determine the poles and residues at each pole of the function $f(z) = \dfrac{z}{z^2-3z+2}$ and hence evaluate $\oint_C f(z)\,dz$ where C is the circle $|z-2|=\frac{1}{2}$. (2011)

Sol. Poles of $f(z)$ are $z^2-3z+2=0$

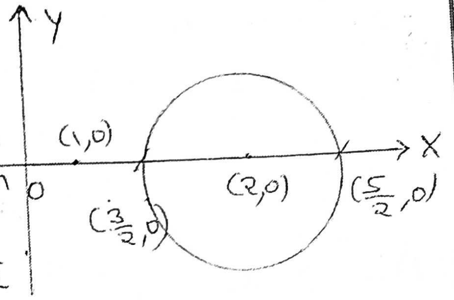
$\Rightarrow (z-1)(z-2)=0 \Rightarrow z=1,2$ (simple poles)

Res. of $f(z)$ at $z=1$ is $R_1 = \lim_{z\to 1} (z-1) f(z) = \lim_{z \to 1} \frac{z}{z-2}$

$$R_1 = \frac{1}{1-2} = -1$$

Res. of $f(z)$ at $z=2$ is $R_2 = \lim_{z\to 2} (z-2) f(z)$

$$R_2 = \lim_{z\to 2} \frac{z}{z-1} = \frac{2}{2-1} = 2$$



Now $C: |z-2| = \frac{1}{2}$ is the circle with centre at 2 i.e point (2,0) and radius $\frac{1}{2}$

clearly only $z=2$ lie inside C and $z=1$ outside C.

Hence by Cauchy Residue theorem

$$\oint_C f(z)\,dz = 2\pi i \, R_2 = 2\pi i\,(2) = 4\pi i$$

Q3. Using Residue theorem, evaluate $\oint_C \frac{z-3}{z^2+2z+5}\,dz$, where C is the circle $|z+1-i| = 2$. (2013)

Sol. The poles of $f(z) = \frac{z-3}{z^2+2z+5}$ are given by

$$z^2+2z+5=0 \Rightarrow z = \frac{-2 \pm \sqrt{4-20}}{2(1)} = -1 \pm 2i$$

C is the circle with centre at $-1+i$ i.e. $(-1,1)$ and radius 2.

The distances of the centre $(-1,1)$ from $z=-1+2i$ i.e. $(-1,2)$ and $z=-1-2i$ i.e. $(-1,-2)$ are respectively 1 and 3.

The first of these is <2 and the second is >2.

Hence only the pole $z=-1+2i$ lies inside C.

(Where $\alpha = -1+2i$)

Residue of $f(z)$ at $z=-1+2i$ is

$$\lim_{z\to -1+2i} (z+1-2i) f(z) = \lim_{z\to\alpha} \frac{(z-\alpha)(z-3)}{z^2+2z+5} \quad \Big| \ \frac{0}{0} \text{ Form}$$

$$= \lim_{z\to\alpha} \frac{(z-\alpha)+(z-3)}{2z+2} \quad \Big| \text{ By L'Hospital's Rule}$$

$$= \frac{\alpha-3}{2\alpha+2} = \frac{-1+2i-3}{-2+4i+2} = \frac{i-2}{2i}$$

Hence by Cauchy's residue theorem

$$\oint_C \frac{z-3}{z^2+2z+5}\,dz = 2\pi i\left(\frac{i-2}{2i}\right) = \pi(i-2) \quad \text{Ans.}$$

Evaluation of Real Integral by using Residue Theorem

Integral of the Type $\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta$

where $F(\sin\theta, \cos\theta)$ is a rational function of $\sin\theta$ and $\cos\theta$.

Put $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta$ i.e. $d\theta = \frac{dz}{iz}$

Also $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$

and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left(z-\frac{1}{z}\right)$

As θ varies from 0 to 2π, z moves once round the unit circle in the anti-clockwise direction.

$$\therefore \int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta = \oint_C F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz}$$

where c is the unit circle $|z|=1$

Q.1. Evaluate by contour integration $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$ where $a>|b|$. Hence or otherwise evaluate

(i) $\int_0^{2\pi} \frac{d\theta}{1-2a\sin\theta+a^2}$, $0<a<1$, (2017)

(ii) $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$ (2011)

Sol. Consider the integration round a unit circle $c: |z|=1$, so that $z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta$

$\Rightarrow d\theta = \frac{dz}{iz}$

Also $\sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) = \frac{1}{2i}\left(z-\frac{1}{z}\right)$

Then the given integral reduces to

$$I = \oint_C \frac{1}{\left[a + \frac{b}{2i}\left(z - \frac{1}{z}\right)\right]} \frac{dz}{iz} = \oint_C \frac{2iz}{2iaz + bz^2 - b} \cdot \frac{dz}{iz}$$

$$= \frac{2}{b} \oint_C \frac{dz}{z^2 + \frac{2ia}{b} z - 1}$$

Poles are given by $z^2 + \frac{2ia}{b} z - 1 = 0$

$$\Rightarrow z = \frac{-\frac{2ia}{b} \pm \sqrt{-\frac{4a^2}{b^2} + 4}}{2} = -\frac{ia}{b} \pm \frac{\sqrt{b^2 - a^2}}{b}$$

$\therefore z = -\frac{ia}{b} \pm i\frac{\sqrt{a^2-b^2}}{b} = \alpha, \beta$ (Simple pole) ($\because a > |b|$)

Where $\alpha = -\frac{ia}{b} + \frac{i\sqrt{a^2-b^2}}{b}$ and $\beta = -\frac{ia}{b} - \frac{i\sqrt{a^2-b^2}}{b}$

clearly $|\beta| > 1$ But $\alpha\beta = -1$

$\therefore |\alpha\beta| = 1 \Rightarrow |\alpha||\beta| = 1 \Rightarrow |\alpha| < 1$

Hence $z = \alpha$ is the only pole which lies inside C

Residue of $f(z)$ at $z = \alpha$ is

$$R = \lim_{z \to \alpha} (z - \alpha) \cdot \frac{2}{b(z-\alpha)(z-\beta)} = \frac{2}{b(\alpha - \beta)} = \frac{2}{b\left(\frac{2i}{b}\sqrt{a^2-b^2}\right)}$$

$$= \frac{1}{i\sqrt{a^2-b^2}}$$

$\therefore$ By Cauchy's Residue theorem

$$I = 2\pi i\,(R) = 2\pi i \frac{1}{i\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}} \quad \text{—— (1)} \quad \text{Ans}$$

$\frac{2\pi}{\sqrt{25-16}}$

Now replace a by $1 + a^2$ and b by $-2a$ in eq (1)

(i)
$$\int_0^{2\pi} \frac{d\theta}{(1+a^2) - 2a\sin\theta} = \frac{2\pi}{\sqrt{(1+a^2)^2 - (-2a)^2}} = \frac{2\pi}{\sqrt{1 + a^4 + 2a^2 - 4a^2}}$$

$$= \frac{2\pi}{\sqrt{1 + a^4 - 2a^2}} = \frac{2\pi}{(1-a^2)} \quad \text{Ans}$$

Also Now replace a by 5 and b by 4 in eq (1)

(ii)
$$\int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta} = \frac{2\pi}{\sqrt{25-16}} = \frac{2\pi}{\sqrt{9}} = \frac{2\pi}{3} \quad \text{Ans}$$

Q2. Using complex integration method, evaluate

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta \qquad [2018, 2012, 2010]$$

Sol.

Let $I = \text{Real part of} \int_0^{2\pi} \frac{e^{2i\theta}}{5+2(e^{i\theta}+e^{-i\theta})}\, d\theta$ | Taking $e^{i\theta}=z$

$= R.P. \text{ of} \oint_C \frac{z^2}{5+2(z+\frac{1}{z})} \frac{dz}{iz}$ | $\therefore d\theta = \frac{dz}{iz}$

$= R.P. \text{ of } \frac{1}{i}\oint_C \frac{z^2}{2z^2+5z+2}\, dz$

Poles are given by $2z^2+5z+2=0 \Rightarrow z = -\frac{1}{2}, -2$

$z=-\frac{1}{2}$ is the only pole which lies inside $C: |z|=1$

Residue of $f(z)$ at $z=-\frac{1}{2}$ is

$$R = \lim_{z\to -\frac{1}{2}} (z+\tfrac{1}{2}) \frac{z^2}{i(2z+1)(z+2)} = \lim_{z\to -\frac{1}{2}} \frac{z^2}{2i(z+2)}$$

$$= \frac{1}{2i}\left(\frac{1}{4}\right)\left(\frac{2}{3}\right) = \frac{1}{12i}$$

Hence by Cauchy's Residue theorem,

$$I = \oint_C f(z)\,dz = 2\pi i\left(\frac{1}{12i}\right) = \frac{\pi}{6}$$ Ans.

Similar Questions

Q3. Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta}\, d\theta$ [2015, 2014, 2007]

(Hint: $\cos 3\theta =$ Real part of $e^{3i\theta}$; Ans $-\frac{\pi}{12}$)

Q4. Evaluate the integral by contour integration

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta . \qquad (2017, 2009)$$

(Ans $\frac{\pi}{12}$)

Examples of the types $\int_0^{\pi} f(\cos\theta, \sin\theta)\, d\theta$ and $\int_{-\pi}^{\pi} f(\cos\theta, \sin\theta)\, d\theta$

Q1. Using Complex integration method, evaluate (2013)

$$\int_0^{\pi} \frac{d\theta}{3+\sin^2\theta}$$

Sol.

Let $I = \int_0^{\pi} \frac{d\theta}{3+\sin^2\theta} = \int_0^{\pi} \frac{d\theta}{3+\frac{1}{2}(1-\cos 2\theta)}$

$= 2\int_0^{\pi} \frac{d\theta}{7-\cos 2\theta}$, Put $2\theta = \phi$, $2\,d\theta = d\phi$

$= \int_0^{2\pi} \frac{d\phi}{7-\cos\phi}$, Put $z = e^{i\phi}$ so that $d\phi = \frac{dz}{iz}$

$= \oint_C \frac{1}{7-\frac{1}{2}(z+\frac{1}{z})} \frac{dz}{iz} = \frac{2}{i}\oint_C \frac{1}{14z-(z^2+1)}\, dz$

$= -\frac{2}{i}\oint_C \frac{dz}{z^2-14z+1}$

Poles are given by $z^2-14z+1=0$

$\Rightarrow z = \frac{14 \pm \sqrt{196-4}}{2(1)} = 7 \pm 4\sqrt{3}$

only $z = 7-4\sqrt{3}$ lies in unit circle $C: |z|=1$

Let $\alpha = 7+4\sqrt{3}$ and $\beta = 7-4\sqrt{3}$

Residue at $z = 7-4\sqrt{3}$ is

$R = \lim_{z\to\beta} (z-\beta)\left(-\frac{2}{i}\right)\frac{1}{(z-\alpha)(z-\beta)}$

$= -\frac{2}{i}\frac{1}{(\beta-\alpha)} = -\frac{2}{i}\left(-\frac{1}{8\sqrt{3}}\right) = \frac{1}{i4\sqrt{3}}$

By Cauchy's Residue theorem

$I = 2\pi i(R) = 2\pi i\left(\frac{1}{i4\sqrt{3}}\right) = \frac{\pi}{2\sqrt{3}}$ Ans.

Q2. Evaluate $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta$

Sol. Let $I = \int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta$

$$= \text{Real Part of } \int_0^{2\pi} \frac{1+2e^{i\theta}}{5+4\cos\theta}\, d\theta$$

Putting $e^{i\theta} = z$ $\therefore d\theta = \frac{dz}{iz}$

$$= \text{R.P. of } \oint_C \frac{1+2z}{5+2(z+\frac{1}{z})} \frac{dz}{iz}$$

$$= \text{R.P. of } \frac{1}{i} \oint_C \frac{1+2z}{2z^2+5z+2}\, dz$$

Poles are given by $2z^2+5z+2=0$

$\Rightarrow (2z+1)(z+2)=0 \Rightarrow z = -\frac{1}{2}, -2$ (simple poles)

Clearly $z = -\frac{1}{2}$ lies inside unit circle $|z|=1$

Residue at $z=-\frac{1}{2}$ is $R = \lim_{z\to -\frac{1}{2}} (z+\frac{1}{2}) \frac{1}{i} \frac{1+2z}{(2z+1)(z+2)}$

$$R = \frac{1}{2i} \lim_{z\to -\frac{1}{2}} \frac{1+2z}{z+2} = 0$$

Hence by Cauchy's Residue theorem

$$I = \int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 2\pi i(0) = 0$$

$$\Rightarrow \int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 0$$

(using property of definite integral

$(\because \cos(2\pi-\theta) = \cos\theta)$

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

Q3. Evaluate the integral $\int_0^{\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\, d\theta$ (2015)

Sol.

Let $I = \int_0^{\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\, d\theta$

($\because \cos 2(\pi-\theta) = \cos(2\pi - 2\theta) = \cos 2\theta$

and $\cos^2 3(\pi-\theta) = [\cos(3\pi - 3\theta)]^2 = \cos^2 3\theta$]

$\therefore \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$, if $f(2a-x) = f(x)$ prop. used

Now $I = \frac{1}{4}\int_0^{2\pi} \frac{1+\cos 6\theta}{5-4\cos 2\theta}\, d\theta$

Consider the integration round a unit circle $c: |z| = 1$

so that $z = e^{i\theta}$ $\therefore dz = ie^{i\theta}d\theta = iz\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$

Also $\cos 2\theta = \frac{1}{2}(e^{2i\theta} + e^{-2i\theta}) = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)$

and $\cos 6\theta = \frac{1}{2}(e^{6i\theta} + e^{-6i\theta}) = \frac{1}{2}\left(z^6 + \frac{1}{z^6}\right)$

Then the given integral reduces to,

$$I = \frac{1}{4}\oint_C \frac{\left[1 + \left(\frac{z^{12}+1}{2z^6}\right)\right]}{5 - 2\left(\frac{z^4+1}{z^2}\right)} \frac{dz}{iz} = -\frac{1}{16i}\oint_C \frac{z^{12}+2z^6+1}{z^5\left(z^4 - \frac{5}{2}z^2+1\right)}\, dz$$

Poles are $z=0$ (order 5) and $z^4 - \frac{5}{2}z^2 + 1 = 0 \Rightarrow$

$\Rightarrow 2z^4 - 5z^2 + 2 = 0 \Rightarrow (2z^2-1)(z^2-2) = 0$

$\Rightarrow z = \pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}$ (simple poles)

Clearly $z=0$ and $z = \pm\frac{1}{\sqrt{2}}$ lie inside c.

Let $f(z) = \frac{z^{12}+2z^6+1}{z^5\left(z^4 - \frac{5}{2}z^2+1\right)}$

$$= \frac{z^{12}+2z^6+1}{z^5}\left[1 - \left(\frac{5}{2}z^2 - z^4\right)\right]^{-1}$$

$$= \frac{(z^6+1)^2}{z^5}\left[1 + \frac{5}{2}z^2 - z^4 + \frac{25}{4}z^4 + z^8 - 5z^6 + \cdots\right]$$

Residue of $f(z)$ at $z=0$ is the coefficient of $\frac{1}{z}$ in this expansion. Hence

$$R_1 = \text{Residue of } f(z) \text{ at } z=0 = -1+\frac{25}{4} = \frac{21}{4}$$

$$R_2 = \text{Residue of } f(z) \text{ at } z=\frac{1}{\sqrt{2}}$$

$$= \lim_{z\to\frac{1}{\sqrt{2}}} \left(z-\frac{1}{\sqrt{2}}\right) \frac{(z^6+1)^2}{z^5(z^2-2)\left(z-\frac{1}{\sqrt{2}}\right)\left(z+\frac{1}{\sqrt{2}}\right)}$$

$$= \lim_{z\to\frac{1}{\sqrt{2}}} \frac{(z^6+1)^2}{z^5(z^2-2)\left(z+\frac{1}{\sqrt{2}}\right)} = -\frac{27}{8}$$

$$R_3 = \text{Residue of } f(z) \text{ at } z=-\frac{1}{\sqrt{2}}$$

$$= \lim_{z\to-\frac{1}{\sqrt{2}}} \left(z+\frac{1}{\sqrt{2}}\right) \frac{(z^6+1)^2}{z^5(z^2-2)\left(z-\frac{1}{\sqrt{2}}\right)\left(z+\frac{1}{\sqrt{2}}\right)}$$

$$= \lim_{z\to-\frac{1}{\sqrt{2}}} \frac{(z^6+1)^2}{z^5(z^2-2)\left(z-\frac{1}{\sqrt{2}}\right)} = -\frac{27}{8}$$

Hence by Cauchy-Residue theorem.

$$I = -\frac{1}{16i}\left[2\pi i\,(R_1+R_2+R_3)\right]$$

$$= -\frac{\pi}{8}\left(\frac{21}{4}-\frac{27}{8}-\frac{27}{8}\right) = -\frac{\pi}{8}\left(\frac{21}{4}-\frac{27}{4}\right) = \frac{3\pi}{16}$$

Ans

Q4. Evaluate the integral $\int_{-\pi}^{\pi} \frac{a\cos\theta}{a+\cos\theta}\,d\theta \; ; a>1$ using complex integration method.

Sol. Let $I = \int_{-\pi}^{\pi} \frac{a\cos\theta}{a+\cos\theta}\, d\theta = 2\int_{0}^{\pi} \frac{a\cos\theta}{a+\cos\theta}\, d\theta$

$I = \int_{0}^{2\pi} \frac{a\cos\theta}{a+\cos\theta}\, d\theta$ (∵ $\cos(-\theta) = \cos\theta$ ∴ $\frac{a\cos\theta}{a+\cos\theta}$ is even fun^c)

(∵ $\cos(2\pi-\theta) = \cos\theta$)

$I = \text{Real Part of } \int_{0}^{2\pi} \frac{a e^{i\theta}}{a+\cos\theta}\, d\theta$ Put $e^{i\theta} = z$ ∴ $d\theta = \frac{dz}{iz}$

$= \text{R.P. of } \oint_C \frac{az}{a+\frac{1}{2}(z+\frac{1}{z})} \frac{dz}{iz}$

$= \text{R.P. of } \frac{2a}{i}\oint_C \frac{z\,dz}{z^2+2az+1}$

Poles are given by

$z^2+2az+1 = 0 \Rightarrow z = \frac{-2a \pm \sqrt{4a^2-4}}{2} = -a \pm \sqrt{a^2-1}$

Only $z = -a+\sqrt{a^2-1}$ lies in $C: |z| = 1$

Let $\alpha = -a+\sqrt{a^2-1}$ and $\beta = -a-\sqrt{a^2-1}$

Residue at $z = -a+\sqrt{a^2-1}$ is

$R = \lim_{z\to\alpha} (z-\alpha)\frac{2a}{i}\frac{z}{(z-\alpha)(z-\beta)}$

$= \lim_{z\to\alpha} \frac{2a}{i}\frac{z}{(z-\beta)} = \frac{2a\alpha}{i(\alpha-\beta)} = \frac{2a(-a+\sqrt{a^2-1})}{i\,2\sqrt{a^2-1}}$

$= \frac{a}{i}\left(-\frac{a}{\sqrt{a^2-1}}+1\right)$

Hence by Cauchy's Residue theorem

$I = 2\pi i(R) = 2\pi i \cdot \frac{a}{i}\left(-\frac{a}{\sqrt{a^2-1}}+1\right)$

$= 2\pi a\left(1-\frac{a}{\sqrt{a^2-1}}\right)$ Ans.